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Linear Toeplitz covariance structure models with optimal estimators of variance components

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Abstract

This paper deals with the problem of optimal quadratic unbiased estimation for statistical models with linear Toeplitz covariance structure. The main result is a necessary and sufficient condition for these models to have an optimal unbiased estimator for any linear combination of variance components. This result is obtained by means of special Jordan algebras which are a powerful tool to characterize optimality in quadratic unbiased estimation.

Keywords: Quadratic unbiased estimation; Special Jordan algebras; Toeplitz matrices; Circulant and skewcirculant matrices

1. Introduction

Autoregressive and moving average models have been largely used for the analysis of time series. For these models, the autocovariance function is known to have a non-linear structure which, for finite sample observations, leads to a symmetric Toeplitz covariance structure. Here are considered finite dimensional models rather than finite sampling of infinite processes. In the same way longitudinal and spatial analyses use finite random vector models rather than infinite processes.

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In this context, it is interesting to consider linear covariance models thus making connection with theory of variance (and covariance) components models. The present framework is therefore to consider a n-random vector Y distributed according to the multivariate quasi-Gaussian distribution with zero expectation and covariance matrix assumed to belong to a linear subspace of the $n \times n$ symmetric Toeplitz matrices linear subspace.

For these types of models, empirical estimators and maximum likelihood estimators have been widely used in the literature [1,3,4,6,27]. Unfortunately, they have no known finite sample properties.

From a statistical point of view, the variance components model approach presented above enables to study their finite sample properties.

This has been done by Marin and Dhorne [18] in the framework of quadratic unbiased estimation. In particular, they showed that there does not exist optimal quadratic unbiased estimators for nested Toeplitz covariance models. On the other hand, they proved the existence of optimal quadratic unbiased estimators for the complete Toeplitz-circulant model.

The main purpose of this paper to investigate the reciprocity of this result, that is to exhibit all the Toeplitz covariance models for which there exists optimal quadratic unbiased estimators.

Section 2 presents the statistical problem and the background on optimal quadratic unbiased estimation of variance components. In Section 3 is presented the algebraical background on optimal estimation. In Section 4 linear Toeplitz covariance structure models are presented.

Section 5 is concerned with the main result of the paper. A characterization of all the Toeplitz covariance structure models for which there exists optimal unbiased estimators is given. In the last section is given the expression of the optimal unbiased estimator of any linear combination of variance components for the complete Toeplitz-circulant and Toeplitz-skewcirculant models.

2. Background on optimality in quadratic unbiased estimation of variance components

Let us consider an observable column random vector $Y = (Y_1 \ \cdots \ Y_n)'$ (u' denotes the transpose of u) distributed according to the multivariate quasi-Gaussian distribution, i.e. the first four moments are equal to those of a multivariate Gaussian distribution, with expectation 0_n (0_n denotes the n-vector with all elements equal to 0) and covariance matrix $\theta_0 I_n + \sum_{i=1}^{r} \theta_i V_i = V_\theta$ (I_n denotes the $n \times n$ identity matrix). The real symmetric matrices $V_1, \ldots, V_r$ are considered as linearly independent matrices. The previous assumption ensures that all linear combinations of $(\theta_0, \ldots, \theta_r)'$ are identifiable. Let us denote $\mathscr{V} = \mathrm{sp}\{I_n, V_1, \ldots, V_r\}$, the linear subspace of the linear space on $\mathbb{R}$ of $n \times n$ real symmetric matrices spanned by the matrices $I_n, V_1, \ldots, V_r$. $\mathscr{V}$ has the dimension $r + 1$. The column vector of unknown covariance parameters $\theta = (\theta_0 \ \cdots \ \theta_r)'$, called variance components, belongs to an open set Θ of $\mathbb{R}^{r+1}$ such that for each $\theta \in \Theta$, V_θ is positive definite. This variance (and covariance) components model is noted model (1).

The statistical problem is the estimation of a linear function $l'\theta$, $l \in \mathbb{R}^{r+1}$, of variance components by a quadratic random variable $Y'AY$; A belongs to the linear space on $\mathbb{R}$ of $n \times n$ real symmetric matrices, such that $\mathbb{E}(Y'AY) = l'\theta$ for all $\theta \in \Theta$ and $\mathbb{V}(Y'AY)$ is minimum (where $\mathbb{E}$ and $\mathbb{V}$ refer, respectively, to expectation and variance operators). It is the concept of MInimum Variance Quadratic Unbiased Estimation (MIVQUE).

The variance of a quadratic random variable in the quasi-Gaussian case is given by $\mathbb{V}(Y'AY) = 2\,\mathrm{Tr}(AV_\theta AV_\theta)$ (where Tr refers to the trace operator). The proof can be found in [25, Chapter 2]. Then, in several cases, the matrix A minimizing $\mathrm{Tr}(AV_\theta AV_\theta)$ under unbiasedness condition depends on θ. In such a case, there just exists Locally MIVQUE (LMIVQUE) of $l'\theta$ for particular choices θ^* on θ.

When the matrix A minimizing $\mathrm{Tr}(AV_\theta AV_\theta)$ under the unbiasedness condition is independent of θ, there exists Uniform MIVQUE (UMIVQUE) of $l'\theta$. A fundamental result due to Seely [26] characterizes the existence of UMIVQUE of $l'\theta$ for any $l \in \mathbb{R}^{r+1}$. In this paper, Seely has shown that there exists a UMIVQUE of $l'\theta$ for any $l \in \mathbb{R}^{r+1}$ if and only if $\mathscr{V}$ is a quadratic subspace.

This algebraic structure is studied in the next section. He has also shown that the symmetric matrix associated to a UMIVQUE is element of $\mathscr{V}$. For additional informations on MIVQUE see [24], [22, Chapter 4], [15,20,23].

If $\mathscr{V}$ is a quadratic subspace, UMIVQUE estimators coincide with maximum likelihood estimators in the Gaussian case. See [19], [23, Chapter 9] for details. As all these concepts coincide, UMIVQUE may be called optimal unbiased estimators. In particular optimal unbiased estimators $\widehat{\theta_i}$ of θ_i can be used to form the $(r+1)$-dimensional statistic $\widehat{\theta} = (\widehat{\theta_0}\ \ \widehat{\theta_1}\ \ \cdots\ \ \widehat{\theta_r})'$. Then for every forms $l'\theta$ one has $\widehat{l'\theta} = l'\widehat{\theta}$. And $\widehat{\theta}$, having similar optimality properties as $l'\widehat{\theta}$, is termed the optimal unbiased estimator of θ. Moreover, Pukelsheim has shown in [21] that the matrix estimate $V_{\widehat{\theta}} = \sum_{i=0}^{r} \widehat{\theta_i} V_i$ for V_θ, obtained from the optimal unbiased estimator $\widehat{\theta}$ of θ is always nonnegative definite.

3. Jordan algebras and quadratic subspaces

In the following, linear spaces, linear subspaces, algebras and Jordan algebras are always defined on the field of real numbers $\mathbb{R}$.

The notion of Jordan algebra being not consistent with usual definition of an algebra, it is necessary to detail some algebraical definitions in the following.

Definition 1. An algebra is a set $\mathscr{A}$ with two internal laws of composition denoted by $+$ and $\times$, and one external law of composition on $\mathbb{R}$ denoted by $\cdot$ such that:

- $(\mathscr{A}, +, \cdot)$ is a linear subspace;
- $\times$ is a commutative law for $\cdot$ and a distributive law for $+$.

The classical definition of an algebra implies also that $\times$ is associative. It is not the case here. If $\times$ is associative, the algebra is called an associative algebra. If $\times$ is commutative, the algebra is called a commutative algebra.

It is now possible to give the formal definition of a Jordan algebra. This notion was initially considered by Jordan, Von Neumann and Wigner in 1934 in a quantum-mechanical context [14]. It is a particular case of non-associative algebras.

Definition 2. A Jordan algebra is a set $\mathscr{A}$ with two internal laws of composition denoted by $+$ and $\times$, and one external law of composition on $\mathbb{R}$ denoted by $\cdot$ such that:

- $(\mathscr{A}, +, \times, \cdot)$ is a commutative algebra;
- $\times$ is a such that $((x \times x) \times y) \times x = (x \times x) \times (y \times x)$ for all x and y element of $\mathscr{A}$.

Let $(\mathscr{A}, +, \times, \cdot)$ be an associative algebra. Consider the induced Jordan law $x * y = \frac{1}{2}(x \times y + y \times x)$ for all x and y element of $\mathscr{A}$. This law of composition is commutative and $((x * x) * y) * x = (x * x) * (y * x)$. Then, if $(\mathscr{J}, +, \cdot)$ is a linear subspace of $(\mathscr{A}, +, \cdot)$ stable through $*$, $(\mathscr{J}, +, *, \cdot)$ is a Jordan algebra. Then is constructed a Jordan (sub-)algebra (not with the law $\times$ but with the induced Jordan law $*$) of an associative algebra.

Definition 3. Let $*$ be the induced Jordan law of $\times$. A Jordan algebra isomorphic to a Jordan (sub-)algebra $(\mathscr{J}, +, *, \cdot)$ of an associative algebra $(\mathscr{A}, +, \times, \cdot)$ is a special Jordan algebra.

In the following, as in previous section, $+$ denotes the usual matrix addition, $\times$ the usual matrix multiplication and $\cdot$ the usual product between a real matrix and a real number. Also, as in the previous section, the matrix product $A \times B$ is denoted by the unadorned matrix product AB. Then, here, $*$ denotes the Jordan matrix product. For two matrices A and B, we have $A * B = \frac{1}{2}(AB + BA)$.

Let $\mathscr{M}_n$ be the set of all $n \times n$ real matrices; $(\mathscr{M}_n, +, \times, \cdot)$ is an associative algebra. Let $\mathscr{S}_n$ be the set of all $n \times n$ real symmetric matrices; $(\mathscr{S}_n, +, *, \cdot)$ is a special Jordan algebra. Also, if $(\mathscr{A}, +, \cdot)$ is a linear subspace of $(\mathscr{S}_n, +, \cdot)$ stable through $*$, then $(\mathscr{A}, +, *, \cdot)$ is a special Jordan algebra.

In the following, only special Jordan algebras of this type are considered and a special Jordan algebra is defined to be a linear subspace of $(\mathscr{S}_n, +, \cdot)$ stable through $*$.

In the following, $\mathscr{A}$ denotes $(\mathscr{A}, +, \cdot)$ or $(\mathscr{A}, +, *, \cdot)$ without confusion.

As was shown in the previous section, Seely in [26] introduces the notion of quadratic subspaces in the context of optimal quadratic unbiased estimation of variance components.

Definition 4. A linear subspace $\mathscr{A}$ of $\mathscr{S}_n$ such that $A \in \mathscr{A}$ implies $A^2 \in \mathscr{A}$ is said to be a quadratic subspace.

The condition $A \in \mathscr{A}$ implies $A^2 \in \mathscr{A}$ for all $A \in \mathscr{A}$ is equivalent to the condition $(A, B) \in \mathscr{A}^2$ implies $\frac{1}{2}(AB + BA) \in \mathscr{A}$ for all $(A, B) \in \mathscr{A}^2$. Then a quadratic subspace is a special Jordan algebra and Seely's result means that there exists an optimal unbiased estimator for any linear combination of variance components if and only if $\mathscr{V}$ is a special Jordan algebra.

Let us now introduce two lemmas used in the following.

Lemma 1. *Let $\mathscr{A}$ be a special Jordan algebra and let $A \in \mathscr{A}$. If $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct eigenvalues of A and $P_1, P_2, \ldots, P_r$ the corresponding $n \times n$ idempotent ($P_i P_i = P_i$) and pairwise orthogonal ($P_i P_j = 0_{(n\times n)}$ ($0_{(n\times n)}$ denotes the $n \times n$ matrix with all elements equal to 0) for $i \neq j$) projection matrices on corresponding eigensubspaces ($A = \sum_{i=1}^{r} \lambda_i P_i$), then $P_1, P_2, \ldots, P_r$ are all element of $\mathscr{A}$.*

The proof can be found in [26].

Let $\mathscr{A}$ be a special Jordan algebra. If for all $(A_1, A_2) \in \mathscr{A}^2$, $A_1 A_2 = A_2 A_1$, then $A_1 * A_2 = A_1 A_2 = A_2 A_1$ and $*$ is associative. We say in this case that $\mathscr{A}$ is an associative special Jordan algebra.

For example, $\mathrm{sp}\{I_n, J_n\}$ (J_n denotes the $n \times n$ matrix with all elements equal to 1) is an associative special Jordan algebra.

Lemma 2. *Let $\mathscr{A}$ be an associative special Jordan algebra of dimension q and $P_1, P_2, \ldots, P_q$ be the $n \times n$ projection matrices on the common eigensubspaces of any matrix which is an element of $\mathscr{A}$. The unique idempotent and pairwise orthogonal basis of $\mathscr{A}$ is $P_1, P_2, \ldots, P_q$.*

Proof. The proof is deduced by considering Lemmas 1 and 6 in [26]. $\square$

For further details on Jordan products and algebras see [12,13,16,17,29].

4. Linear Toeplitz covariance structure models

A symmetric Toeplitz matrix is an $n \times n$ matrix T with elements (i, j), $t_{i,j}$, such that, for all $(i, j) \in \{1, \ldots, n\}^2$, $t_{i,j} = t_{|i-j|}$, i.e. a matrix of the form

$$T = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{n-1} \\ t_1 & t_0 & t_1 & \cdots & t_{n-2} \\ t_2 & t_1 & t_0 & \cdots & t_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{bmatrix} = \mathrm{Toep}(t_0, t_1, \ldots, t_{n-1}).$$

A great deal is known about the behavior of such matrices. The most common and complete reference is the book written by Grenander and Szegö [8]. More recent texts devoted to the subject are the sixth version of the unpublished document written by Gray [7] and the book written by Böttcher and Silbermann [2].

Let us denote by $\mathscr{B}_n$ the set of $n \times n$ symmetric Toeplitz matrices.

Notice that

$$\begin{aligned}&\mathrm{Toep}(a_0, \dots, a_{n-1}) + \alpha \mathrm{Toep}(b_0, \dots, b_{n-1})\\ &\quad = \mathrm{Toep}(a_0 + \alpha b_0, \dots, a_{n-1} + \alpha b_{n-1})\end{aligned}$$

so that $\mathscr{B}_n$ forms a linear subspace of $\mathscr{S}_n$.

Moreover, for all $n \geqslant 2$ and $k \in \{1, \dots, n-1\}$, let us define

$$B_{n,k} = \mathrm{Toep}\left(\overbrace{\underbrace{0, \dots, 0}_{k}, 1, 0, \dots, 0}^{n}\right).$$

$B_{n,k}$ has components (i, j) equal to one when $|i - j| = k$ and equal to zero otherwise. The matrix $B_{n,k}$ is called the $n \times n$ symmetric Toeplitz matrix of order k.

Notice that $\mathrm{Toep}(t_0, \dots, t_{n-1}) = t_0 I_n + \sum_{i=1}^{n-1} t_i B_{n,i}$ and that $\alpha_0 I_n + \sum_{i=1}^{n-1} \alpha_i \times B_{n,i} = 0 I_n$ implies that $(\alpha_0\ \alpha_1\ \dots\ \alpha_{n-1})' = 0_n$. Then, $\mathscr{B}_n = \mathrm{sp}\{I_n, B_{n,1}, \dots, B_{n,n-1}\}$ (the linear subspace of $\mathscr{S}_n$ spanned by the matrices $I_n, B_{n,1}, \dots, B_{n,n-1}$) and $I_n, B_{n,1}, \dots, B_{n,n-1}$ is a basis of $\mathscr{B}_n$.

Linear Toeplitz covariance structure models are some specific cases of model (1). Additional assumptions that $V_i \in \mathscr{B}_n$ for all $i \in \{1, \dots, r\}$ are needed. This implies that the covariance matrix of the observed random vector Y belongs to a linear subspace of $\mathscr{B}_n$ containing I_n. Also, the identity matrix I_n is always taken in the basis of this linear subspace which specifies the parametrization chosen for the model.

For example, the model for which $r = 1$ and $V_1 = J_n$ is a linear Toeplitz covariance structure model because $J_n \in \mathscr{B}_n$. The first variance component θ_0 is associated with the identity matrix and the second variance component θ_1 is associated with the matrix J_n. The model for which $r = 1$ and $V_1 = \sum_{i=1}^{n-1} B_{n,i}$ is obviously the same model than the previous ($\sum_{i=1}^{n-1} B_{n,i} = J_n - I_n$). But it corresponds to an other parametrization. The second variance component θ_1 is here associated to the matrix $\sum_{i=1}^{n-1} B_{n,i}$.

The linear Toeplitz covariance structure models for which $V_i = B_{n,i}$ for all $i \in \{1, \dots, r\}$ is called the nested Toeplitz model of order r. It is, for a random vector, the linear form of a moving average process of order r.

Two linear Toeplitz covariance structure models of interest in the following are presented.

For all $n \geqslant 3$, a symmetric circulant matrix is an $n \times n$ matrix C of the form:

$$\begin{aligned}&\text{if } n \text{ is odd, } C = \mathrm{Toep}(c_0, c_1, \dots, c_{(n-1)/2}, c_{(n-1)/2}, \dots, c_1),\\ &\text{if } n \text{ is even, } C = \mathrm{Toep}(c_0, c_1, \dots, c_{n/2}, c_{(n/2)-1}, \dots, c_1).\end{aligned}$$

The properties of circulant matrices are well known and easily derived. The most common and complete reference is the book written by Davis [5].

Let us denote by $\mathscr{C}_n$ the set of $n \times n$ symmetric circulant matrices and let $[w]_e$ denotes the greatest integer not exceeding $w \geqslant 0$.

Notice that, if n is odd,

$$\begin{aligned}&\mathrm{Toep}(a_0, a_1, \dots, a_{(n-1)/2}, a_{(n-1)/2}, \dots, a_1)\\ &\quad + \alpha \mathrm{Toep}(b_0, b_1, \dots, b_{(n-1)/2}, b_{(n-1)/2}, \dots, b_1)\\ &\qquad = \mathrm{Toep}\big(a_0 + \alpha b_0, a_1 + \alpha b_1, \dots, a_{(n-1)/2} + \alpha b_{(n-1)/2},\\ &\qquad\qquad a_{(n-1)/2} + \alpha b_{(n-1)/2}, \dots, a_1 + \alpha b_1\big)\end{aligned}$$

and we have the same equality if n is even. Then, $\mathscr{C}_n$ forms a linear subspace of $\mathscr{S}_n$.

Moreover, for all $n \geqslant 3$ and $k \in \{1, \dots, [(n-1)/2]_e\}$, let us define

$$C_{n,k} = \mathrm{Toep}\left(\overbrace{\underbrace{0, \dots, 0}_{k}, 1, 0, \dots, 0, 1, \underbrace{0, \dots, 0}_{k-1}}^{n}\right)$$

and if n is even $C_{n,(n/2)} = B_{n,(n/2)}$.

The matrix $C_{n,k}$ is called the $n \times n$ symmetric circulant matrix of order k. Let us define the $n \times n$ matrix

$$L_n = \left[\begin{array}{c|c} 0_n & I_{n-1} \\ \hline 1 & 0'_n \end{array}\right].$$

L_n is a permutation matrix which plays a fundamental role in the theory of circulant matrices. Actually, a second representation of the $n \times n$ symmetric circulant matrix of order k is $C_{n,k} = (L_n)^k + (L_n)^{n-k}$ for all $k \in \{1, \dots, [(n-1)/2]_e\}$ and $C_{n,(n/2)} = (L_n)^{n/2}$ if n is even. Refer to [5, p. 68].

Let us introduce one lemma on the linear subspace $\mathscr{C}_n$.

Lemma 3. *For all $n \geqslant 3$, $\mathscr{C}_n$ is a special Jordan algebra.*

Proof. Davis in [5, p. 74, Theorem 3.2.4] has shown that if A and B are circulant matrices then $AB = BA$ and AB is a circulant matrix.

Therefore, for all $(A, B) \in \mathscr{C}_n \times \mathscr{C}_n$, we have $AB = BA$ and AB is a circulant matrix. As $(AB)' = B'A' = BA$, for all $(A, B) \in \mathscr{C}_n \times \mathscr{C}_n$, we have $AB = BA$ and $AB \in \mathscr{C}_n$. Then, for all $(A, B) \in \mathscr{C}_n \times \mathscr{C}_n$, $A * B = AB = BA \in \mathscr{C}_n$ and $\mathscr{C}_n$ is an associative special Jordan algebra. $\square$

For $(i, j) \in \{1, \dots, [(n-1)/2]_e\}^2$, we have:

$$\begin{aligned}C_{n,i} C_{n,j} &= \big((L_n)^i + (L_n)^{n-i}\big)\big((L_n)^j + (L_n)^{n-j}\big)\\ &= (L_n)^{i+j} + (L_n)^{n-j+i} + (L_n)^{n-i+j} + (L_n)^{2n-i-j}\end{aligned}$$

Moreover $(L_n)^n = I_n$, you can see in [5, p. 27]. Then

$$(L_n)^{2n-i-j} = (L_n)^{n-i-j},\ \ (L_n)^{n-j+i} = (L_n)^{i-j},\ \ (L_n)^{n-i+j} = (L_n)^{j-i}.$$

Therefore, for all $(i, j) \in \{1, \ldots, [(n-1)/2]_e\}$, we have

- if $i + j < (n/2)$ and $i \neq j$,

$$C_{n,i}C_{n,j} = C_{n,i+j} + C_{n,|i-j|}; \tag{1}$$

- if $i + j < (n/2)$ and $i = j$,

$$C_{n,i}C_{n,j} = C_{n,i+j} + 2I_n; \tag{2}$$

- if $i + j > (n/2)$ and $i \neq j$,

$$C_{n,i}C_{n,j} = C_{n,n-i-j} + C_{n,|i-j|}; \tag{3}$$

- if $i + j > (n/2)$ and $i = j$,

$$C_{n,i}C_{n,j} = C_{n,n-i-j} + 2I_n. \tag{4}$$

Also if n is even, $i + j = (n/2)$ and $i \neq j$,

$$C_{n,i}C_{n,j} = 2C_{n,(n/2)} + C_{n,|i-j|}. \tag{5}$$

If $(n/2)$ is even,

$$C_{n,(n/2)}C_{n,(n/2)} = I_n \quad \text{and} \quad C_{n,(n/4)}C_{n,(n/4)} = 2C_{n,(n/2)} + 2I_n. \tag{6}$$

If n is even and $j \in \{1, \ldots, [(n-1)/2]_e\}$,

$$C_{n,(n/2)}C_{n,j} = C_{n,j}C_{n,(n/2)} = C_{n,((n/2)-j)}. \tag{7}$$

All these equalities are used to show Proposition 4 in Section 6.

Notice that

$$\mathrm{Toep}(c_0, c_1, c_2, \ldots, c_1) = c_0 I_n + \sum_{i=1}^{[n/2]_e} c_i C_{n,i}$$

and that

$$\alpha_0 I_n + \sum_{i=1}^{[n/2]_e} \alpha_i C_{n,i} = 0 I_n$$

implies that $(\alpha_0 \ \alpha_1 \ \ldots \ \alpha_{[n/2]_e})' = 0_{[n/2]_e+1}$.
Then $\mathscr{C}_n = \mathrm{sp}\{I_n, C_{n,1}, \ldots, C_{n,[n/2]_e}\}$ and $I_n, C_{n,1}, \ldots, C_{n,[n/2]_e}$ is a basis of $\mathscr{C}_n$. The dimension of $\mathscr{C}_n$ is $[n/2]_e + 1$.

Moreover for all $k \in \{1, \ldots, [(n-1)/2]_e\}$ we have $C_{n,k} = B_{n,k} + B_{n,n-k}$ and $C_{n,(n/2)} = B_{n,(n/2)}$ if n is even. Thus the model for which $r = [n/2]_e$ and $V_i = C_{n,i}$ for all $i \in \{1, \ldots, [n/2]_e\}$ is a linear Toeplitz covariance structure model because $C_{n,i} \in \mathscr{B}_n$ for all $i \in \{1, \ldots, [n/2]_e\}$. This model is called the complete Toeplitz-circulant model.

For all $n \geqslant 3$, a symmetric skewcirculant matrix is an $n \times n$ matrix D of the form:

$$\begin{aligned} &\text{if } n \text{ is odd,} \quad D = \mathrm{Toep}\left(d_0, d_1, \ldots, d_{(n-1)/2}, -d_{(n-1)/2}, \ldots, -d_1\right) \\ &\text{if } n \text{ is even,} \quad D = \mathrm{Toep}\left(d_0, d_1, \ldots, d_{(n/2)-1}, 0, -d_{(n/2)-1}, \ldots, -d_1\right). \end{aligned}$$

The properties of skewcirculant matrices are derived in [5].

Let us denote by $\mathscr{D}_n$ the set of $n \times n$ symmetric circulant matrices.

Notice that, if n is odd,

$$\begin{aligned}
&\mathrm{Toep}\left(a_0, a_1, \ldots, a_{(n-1)/2}, -a_{(n-1)/2}, \ldots, -a_1\right) \\
&\quad + \alpha \mathrm{Toep}\left(b_0, b_1, \ldots, b_{(n-1)/2}, -b_{(n-1)/2}, \ldots, -b_1\right) \\
&\qquad = \mathrm{Toep}\left(a_0 + \alpha b_0, a_1 + \alpha b_1, \ldots, a_{(n-1)/2} + \alpha b_{(n-1)/2},\right. \\
&\qquad\qquad \left. - a_{(n-1)/2} - \alpha b_{(n-1)/2}, \ldots, -a_1 - \alpha b_1\right)
\end{aligned}$$

and we have the same equality if n is even. Then, $\mathscr{D}_n$ forms a linear subspace of $\mathscr{S}_n$.

Moreover, for all $n \geqslant 3$ and $k \in \{1, \ldots, [(n-1)/2]_e\}$, let us define

$$D_{n,k} = \mathrm{Toep}\left(\overbrace{\underbrace{0, \ldots, 0}_{k}, 1, 0, \ldots, 0, -1, \underbrace{0, \ldots, 0}_{k-1}}^{n}\right).$$

The matrix $D_{n,k}$ is called the $n \times n$ symmetric skewcirculant matrix of order k. Let us define the $n \times n$ matrix

$$M_n = \left[\begin{array}{c|c} 0_n & I_{n-1} \\ \hline -1 & 0_n' \end{array}\right].$$

A second representation of the $n \times n$ symmetric skewcirculant matrix of order k is $D_{n,k} = (M_n)^k - (M_n)^{n-k}$. Refer to [5, p. 84].

Let us introduce one lemma on the linear subspace $\mathscr{D}_n$.

Lemma 4. *For all $n \geqslant 3$, $\mathscr{D}_n$ is a special Jordan algebra.*

Proof. Davis in [5, p. 85] has shown that if A and B are skewcirculant matrices then $AB = BA$ and AB is a skewcirculant matrix. Therefore, a same reasoning as for $\mathscr{C}_n$ can be used and $\mathscr{D}_n$ is an associative special Jordan algebra. □

For all $(i, j) \in \{1, \ldots, [(n-1)/2]_e\}$, we have

$$\begin{aligned}
D_{n,i} D_{n,j} &= \left((M_n)^i - (M_n)^{n-i}\right)\left((M_n)^j - (M_n)^{n-j}\right) \\
&= (M_n)^{i+j} - (M_n)^{n-j+i} - (M_n)^{n-i+j} + (M_n)^{2n-i-j}.
\end{aligned}$$

Moreover $(M_n)^n = -I_n$, you can see in [5, p. 85].

Then

$$\begin{aligned}
&(M_n)^{2n-i-j} = -(M_n)^{n-i-j}, \\
&(M_n)^{n-j+i} = -(M_n)^{i-j}, \\
&(M_n)^{n-i+j} = -(M_n)^{j-i}.
\end{aligned}$$

Therefore, for all $(i, j) \in \{1, \ldots, [(n-1)/2]_e\}$, we have:

- if $i + j < (n/2)$ and $i \neq j$,

$$D_{n,i} D_{n,j} = D_{n,i+j} + D_{n,|i-j|}; \tag{8}$$

- if $i + j < (n/2)$ and $i = j$,

$$D_{n,i} D_{n,j} = D_{n,i+j} + 2I_n; \tag{9}$$

- if $i + j > (n/2)$ and $i \neq j$,

$$D_{n,i} D_{n,j} = -D_{n,n-i-j} + D_{n,|i-j|}; \tag{10}$$

- if $i + j > (n/2)$ and $i = j$,

$$D_{n,i} D_{n,j} = -D_{n,n-i-j} + 2I_n. \tag{11}$$

Also if n is even, $i + j = (n/2)$ and $i \neq j$

$$D_{n,i} D_{n,j} = D_{n,|i-j|}. \tag{12}$$

If $(n/2)$ is even,

$$D_{n,(n/4)} D_{n,(n/4)} = 2I_n. \tag{13}$$

All these equalities are used to show Proposition 4 in Section 6.

Notice that

$$\mathrm{Toep}(d_0, d_1, d_2, \ldots, -d_1) = d_0 I_n + \sum_{i=1}^{[(n-1)/2]_e} d_i D_{n,i}$$

and that

$$\alpha_0 I_n + \sum_{i=1}^{[(n-1)/2]_e} \alpha_i D_{n,i} = 0 I_n$$

implies that $(\alpha_0 \ \ \alpha_1 \ \ \ldots \ \ \alpha_{[(n-1)/2]_e})' = 0_{[(n-1)/2]_e+1}$.

Then $\mathscr{D}_n = \mathrm{sp}\{I_n, D_{n,1}, \ldots, D_{n,[(n-1)/2]_e}\}$ and $I_n, D_{n,1}, \ldots, D_{n,[(n-1)/2]_e}$ is a basis of $\mathscr{D}_n$. The dimension of $\mathscr{D}_n$ is $[(n-1)/2]_e + 1$.

Moreover, for all $k \in \{1, \ldots, [(n-1)/2]_e\}$, we have $D_{n,k} = B_{n,k} - B_{n,n-k}$. Thus the model for which $r = [(n-1)/2]_e$ and $V_i = D_{n,i}$ for all $i \in \{1, \ldots, [(n-1)/2]_e\}$ is a linear Toeplitz covariance structure model because $D_{n,i} \in \mathscr{B}_n$ for all $i \in \{1, \ldots, [(n-1)/2]_e\}$. This model is called the complete Toeplitz-skewcirculant model.

5. Toeplitz covariance structure models for which there exists optimal unbiased estimators

In this section, we look for all linear Toeplitz covariance structure models for which there exists an optimal unbiased estimator for any linear combination of variance components. That is all linear subspaces of $\mathscr{B}_n$ containing the identity matrix which have a special Jordan algebra structure.

If $n = 2$, as $B_{2,1} * B_{2,1} = 2I_2$, there exists optimal unbiased estimators for all linear Toeplitz covariance structure models.

Here are studied the other cases.

Let us introduce the central following lemma.

Lemma 5. *Let $A \in \mathscr{B}_n$. If $A^2 \in \mathscr{B}_n$, then $A \in \mathscr{C}_n$ or $A \in \mathscr{D}_n$.*

Proof. Let $A = a_0 I_n + \sum_{i=1}^{n-1} a_i B_{n,i}$, and for all $i \in \{0, \ldots, n-1\}$, let us define the column vector of $\mathbb{R}^n$ $A_i = (a_i \quad a_{i-1} \cdots a_0 \cdots a_{n-i-1})'$.

We have

$$A = \begin{bmatrix} A_0' \\ - \\ \vdots \\ - \\ A_{n-1}' \end{bmatrix}$$

and $A = [A_0 | \cdots | A_{n-1}]$.

$$A_{i+1}' A_{i+1} = A_i' A_i + a_{i+1}^2 - a_{n-i-1}^2 \quad \text{for all } i \in \{0, \ldots, n-2\}.$$

Then, all the diagonal terms of A^2 are equal if $a_{i+1}^2 = a_{n-i-1}^2$ for all $i \in \{0, \ldots, n-2\}$.

Moreover,

$$A_{i+1}' A_{i+1+j} = A_i' A_{i+j} + a_{i+1} a_{i+1+j} - a_{n-i-1} a_{n-i-1-j} \quad \text{for all } i \in \{0, \ldots, n-2-j\}$$

and for all $j \in \{1, \ldots, n-2\}$. Then all the terms which are elements of the jth upper-stripe and which are elements of the jth lower-stripe of A^2 are equal if $a_{i+1} a_{i+1+j} = a_{n-i-1} a_{n-i-1-j}$ for all $i \in \{0, \ldots, n-2-j\}$.

Therefore, A^2 is a symmetric Toeplitz matrix if $a_i a_j = a_{n-i} a_{n-j}$ for all $(i, j) \in \{1, \ldots, n-1\}^2$. Then, A^2 is a symmetric Toeplitz matrix if ($a_i = a_{n-i}$ for all $i \in \{1, \ldots, [n/2]_e\}$) or ($a_i = -a_{n-i}$ for all $i \in \{1, \ldots, [(n-1)/2]_e\}$ and $a_{n/2} = 0$ if n is even). □

Note that this lemma is a corollary to a result proved by Zimmerman in [28]. Let $(A, B) \in \mathscr{B}_n \times \mathscr{B}_n$. Zimmerman has shown that $AB \in \mathscr{B}_n$ if and only if (A or B is diagonal) or (A and B are both circulant or skewcirculant matrices). In the same field, Ikramov and Chugunov in [11] have studied the skew-symmetric part of the product of Toeplitz matrices.

Let $A \in \mathscr{B}_n$. By considering Lemmas 3–5, the matrix A is an element of a special Jordan algebra $\mathscr{A}$ such that $I_n \in \mathscr{A}$ and $\mathscr{A} \subseteqq \mathscr{B}_n$ if and only if $A \in \mathscr{C}_n$ or $A \in \mathscr{D}_n$.

Note that this result can be shown in another way. Let A be a nonsingular Toeplitz matrix, by generalizing a result proved by Huang and Cline in [10], Greville has shown in [9] that A^{-1} is a Toeplitz matrix if and only if A is a k-circulant matrix.

The k-circulant matrix A is such that

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ ka_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ ka_{n-2} & ka_{n-1} & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ka_1 & ka_2 & ka_3 & \dots & a_0 \end{bmatrix}.$$

The inverse of a symmetric matrix is itself symmetric and a k-circulant matrix is symmetric if and only if it is a symmetric circulant or a symmetric skewcirculant matrix. Therefore, let $A \in \mathscr{B}_n$ and A nonsingular, $A^{-1} \in \mathscr{B}_n$ if and only if A is a symmetric circulant or symmetric skewcirculant matrix. Then any subclass of nonsingular symmetric Toeplitz matrices which is closed with respect to inversion is contained in either $\mathscr{C}_n$ or $\mathscr{D}_n$.

Let $\mathscr{U}$ be a linear subspace of $\mathscr{S}_n$ containing I_n. Jensen has shown in [13] that for all $A \in \mathscr{U}$ such that A is nonsingular, $A^{-1} \in \mathscr{U}$ if and only if $\mathscr{U}$ is a special Jordan algebra.

Let $A \in \mathscr{B}_n$, this implies that the matrix A is an element of a special Jordan algebra $\mathscr{A}$ such that $I_n \in \mathscr{A}$ and $\mathscr{A} \subseteqq \mathscr{B}_n$ if and only if $A \in \mathscr{C}_n$ or $A \in \mathscr{D}_n$.

Therefore, the linear Toeplitz covariance structure models for which there exists optimal unbiased estimators are composed exclusively of symmetric circulant matrices or exclusively of symmetric skewcirculant matrices.

Since $\mathscr{C}_n$ is an associative special Jordan algebra and by considering Lemmas 1 and 2, it is clear that all submodels of the complete Toeplitz-circulant model for which there exists optimal unbiased estimators are composed of the identity matrix and of any subset of linear combination of a pairwise orthogonal basis of $\mathscr{C}_n$.

The following proposition gives the unique idempotent and pairwise orthogonal basis of $\mathscr{C}_n$.

For all $n \geqslant 3$ and for all $s \in \{1, \dots, [(n-1)/2]_e\}$, let us define the $n \times n$ matrix

$$R_s = \frac{2}{n}\left(I_n + \sum_{i=1}^{[n/2]_e} \cos\left(\frac{2\pi i s}{n}\right) C_{n,i}\right).$$

Proposition 1. *The unique idempotent and pairwise orthogonal basis of* $\mathscr{C}_n$ *is*

- *if n is odd*, $(1/n)J_n, R_1, \dots, R_{(n-1)/2}$;
- *if n is even*, $(1/n)J_n, R_1, \dots, R_{(n/2)-1}, \frac{1}{n}(I_n + \sum_{i=1}^{n/2}(-1)^i C_{n,i})$.

Proof. By using Lemma 2, we have just to calculate the projection matrices on the common eigensubspaces of any matrix element of $\mathscr{C}_n$. We choice the matrix $C_{n,1} = L_n + (L_n)^{n-1} = L_n + (L_n)^{-1}$. Davis has shown in [5, p. 72 (Theorem 3.2.1)] that the matrix L_n has n distinct eigenvalues given by $\lambda_s = \exp(\mathrm{i}2\pi s/n)$ for all $s \in \{0, \dots, n-1\}$ and that an column eigenvector of L_n corresponding to the eigenvalue λ_s is $v_s = (1\ \exp(\mathrm{i}2\pi s/n)\ \exp(\mathrm{i}4\pi s/n)\ \dots\ \exp(\mathrm{i}2(n-1)\pi s/n))'$.

Moreover,

$$\begin{aligned}C_{n,1}v_s &= (L_n + (L_n)^{-1})v_s\\ &= L_n v_s + (L_n)^{-1}v_s\\ &= \exp\left(\frac{\mathrm{i}2\pi s}{n}\right)v_s + \exp\left(\frac{-\mathrm{i}2\pi s}{n}\right)v_s\\ &= 2\cos\left(\frac{2\pi s}{n}\right)v_s.\end{aligned}$$

If $s \in \{1, \ldots, [(n-1)/2]_e\}$ we have $2\cos(2\pi s/n) = 2\cos(2\pi(n-s)/n)$ then $2\cos(2\pi s/n)$ is a double eigenvalues of $C_{n,1}$. Corresponding to this double eigenvalue, we can take the characteristic vectors $u_{s,1} = (1/2)(v_s + v_{n-s})$ and $u_{s,2} = (1/2\mathrm{i})(v_s - v_{n-s})$. Thus the eigensubspace corresponding to the eigenvalue $2\cos(2\pi 0/n) = 2$ is

$$\mathrm{sp}\{v_0\} = \mathrm{sp}\left\{\overbrace{(1 \quad \cdots \quad 1)'}^{n}\right\}; \quad \text{for all } s \in \{1, \ldots, [(n-1)/2]_e\}$$

the eigensubspace corresponding to the eigenvalue $2\cos(2\pi s/n)$ is $\mathrm{sp}\{u_{s,1}, u_{s,2}\}$; and if n is even the eigensubspace corresponding to the eigenvalue $2\cos(2\pi n/2n) = -2$ is

$$\mathrm{sp}\{v_{n/2}\} = \mathrm{sp}\left\{\overbrace{(1 \quad -1 \quad \cdots \quad 1 \quad -1)'}^{n}\right\}.$$

For all $s \in \{1, \ldots, [(n-1)/2]_e\}$, we have

$$u_{s,1} = \left(1 \quad \cos\left(\frac{2\pi s}{n}\right) \quad \cos\left(\frac{4\pi s}{n}\right) \quad \cdots \quad \cos\left(\frac{2(n-1)\pi s}{n}\right)\right)'$$

and

$$u_{s,2} = \left(0 \quad \sin\left(\frac{2\pi s}{n}\right) \quad \sin\left(\frac{4\pi s}{n}\right) \quad \cdots \quad \sin\left(\frac{2(n-1)\pi s}{n}\right)\right)'.$$

Then, the projection matrix on the eigensubspace corresponding to the eigenvalue 2 is $(1/n)J_n$; for all $s \in \{1, \ldots, [(n-1)/2]_e\}$ the projection matrix on the eigensubspace corresponding to the eigenvalue $2\cos(2\pi s/n)$ is R_s; and finally if n is even the projection matrix on the eigensubspace corresponding to the eigenvalue -2 is $(1/n)(I_n + \sum_{i=1}^{n/2}(-1)^i C_{n,i})$. $\square$

Since for the complete Toeplitz-circulant model, it is clear that all submodels of the complete Toeplitz-skewcirculant model for which there exists optimal unbiased estimators are composed of the identity matrix and of any subset of linear combination of a pairwise orthogonal basis of $\mathscr{D}_n$. The following proposition gives the unique idempotent and pairwise orthogonal basis of $\mathscr{D}_n$.

For all $n \geqslant 3$ and for all $s \in \{1, \dots, [n/2]_e\}$, let us define the $n \times n$ matrix

$$T_s = \frac{2}{n}\left(I_n + \sum_{i=1}^{[(n-1)/2]_e} \cos\left(\frac{\pi i(2s-1)}{n}\right) D_{n,i}\right).$$

Proposition 2. *For all $n \geqslant 3$, the unique idempotent and pairwise orthogonal basis of $\mathscr{D}_n$ is*
- *if n is even, $T_1, \dots, T_{n/2}$;*
- *if n is odd, $T_1, \dots, T_{(n-1)/2}$, $\frac{1}{n}(I_n + \sum_{i=1}^{(n-1)/2}(-1)^i D_{n,i})$.*

Proof. By using Lemma 1, we have just to calculate the projection matrices on the common eigensubspaces of any matrix element of $\mathscr{D}_n$. We choice the matrix $D_{n,1} = M_n - (M_n)^{n-1} = M_n + (M_n)^{-1}$. Davis has shown in [5, p. 84] that the matrix M_n has n distinct eigenvalues given by $\lambda_s = \exp(\mathrm{i}\pi(2s-1)/n)$ for all $s \in \{1, \dots, n\}$ and that an column eigenvector of M_n corresponding to the eigenvalue λ_s is

$$v_s = \left(1 \quad \exp\left(\frac{\mathrm{i}\pi(2s-1)}{n}\right) \quad \exp\left(\frac{\mathrm{i}2\pi(2s-1)}{n}\right) \right.$$
$$\left. \cdots \quad \exp\left(\frac{\mathrm{i}(n-1)\pi(2s-1)}{n}\right)\right)'.$$

Moreover,

$$\begin{aligned} D_{n,1}v_s &= \big(M_n + (M_n)^{-1}\big)v_s \\ &= M_n v_s + (M_n)^{-1}v_s \\ &= \exp\left(\frac{\mathrm{i}\pi(2s-1)}{n}\right)v_s + \exp\left(\frac{-\mathrm{i}\pi(2s-1)}{n}\right)v_s \\ &= 2\cos\left(\frac{\pi(2s-1)}{n}\right)v_s \end{aligned}$$

If $s \in \{1, \dots, [n/2]_e\}$ we have $2\cos(\pi(2s-1)/n) = 2\cos(\pi(2(n-s)-1)/n)$, then $2\cos(\pi(2s-1)/n)$ is a double eigenvalues of $D_{n,1}$. Corresponding to this double eigenvalue, we can take the characteristic vectors $u_{s,1} = (1/2)(v_s + v_{n-s})$ and $u_{s,2} = (1/2\mathrm{i})(v_s - v_{n-s})$.

Thus, for all $s \in \{1, \dots, [n/2]_e\}$, the eigensubspace corresponding to the eigenvalue $2\cos(\pi(2s-1)/n)$ is $\mathrm{sp}\{u_{s,1}, u_{s,2}\}$ and if n is odd the eigensubspace corresponding to the eigenvalue $2\cos(\pi(2(n+1/2)-1)n) = -2$ is

$$\mathrm{sp}\{v_{(n+1)/2}\} = \mathrm{sp}\left\{\overbrace{(-1 \quad 1 \quad \cdots \quad -1 \quad 1 \quad -1)}^{n}{}'\right\}.$$

For all $s \in \{1, \ldots, [n/2]_e\}$, we have

$$u_{s,1} = \left(1 \quad \cos\left(\frac{\pi(2s-1)}{n}\right) \quad \cos\left(\frac{\pi 2(2s-1)}{n}\right) \right.$$
$$\left. \cdots \quad \cos\left(\frac{\pi(n-1)(2s-1)}{n}\right)\right)'$$

and

$$u_{s,2} = \left(0 \quad \sin\left(\frac{\pi(2s-1)}{n}\right) \quad \sin\left(\frac{\pi 2(2s-1)}{n}\right) \right.$$
$$\left. \cdots \quad \sin\left(\frac{\pi(n-1)(2s-1)}{n}\right)\right)'.$$

Then the projection matrix on the eigensubspace corresponding to the eigenvalue $2\cos(\pi(2s-1)/n)$ is R_s and finally if n is odd the projection matrix on the eigensubspace corresponding to the eigenvalue -2 is $(1/n)(I_n + \sum_{i=1}^{(n-1)/2}(-1)^i D_{n,i})$. □

6. Optimal unbiased estimators for the complete Toeplitz-circulant and Toeplitz-skewcirculant models

The following two propositions give the expressions of the optimal unbiased estimator of any linear combination of variance components for the complete Toeplitz-circulant and Toeplitz-skewcirculant models.

Proposition 3. *For the complete Toeplitz-circulant model, the optimal unbiased estimator of $l'\theta$ for all $l \in \mathbb{R}^{r+1}$ is*

$$\widehat{l'\theta} = \frac{l_0}{n}\sum_{j=1}^{n} Y_j^2 + \sum_{i=0}^{[n/2]_e} \frac{l_i}{n}\left(\sum_{j=1}^{n-i} Y_j Y_{j+i} + \sum_{j=0}^{i-1} Y_{i-j} Y_{n-j}\right). \tag{14}$$

Proof. For complete Toeplitz-circulant model, the optimal unbiased estimator $Y'AY$ of $l'\theta$ is such that $A = \lambda_0 I_n + \sum_{i=1}^{[n/2]_e} \lambda_i C_{n,i}$ ($\mathscr{C}_n$ is a special Jordan algebra). Moreover

$$\mathbb{E}(Y'AY) = \operatorname{Tr}\left(A\left(\theta_0 I_n + \sum_{i=1}^{[n/2]_e} \theta_i C_{n,i}\right)\right).$$

This is a standard result the proof can be found in [25, Chapter 2] for example. Then $\mathbb{E}(Y'AY) = l'\theta$ for all $\theta \in \Theta$ if and only if $\operatorname{Tr}(A) = l_0$ and, for all $i \in \{1, \ldots, [n/2]_e\}$, $\operatorname{Tr}(AC_{n,i}) = l_i$. By considering equalities (1)–(7), the unbiased condition leads to

$\lambda_0 = l_0/n$, for all $i \in \{1, \ldots, [(n-1)/2]_e\}$, $\lambda_i = l_i/2n$ and, if n is even, $\lambda_{n/2} = l_{n/2}/n$.

Moreover, $Y'V_0Y = Y'I_nY = \sum_{j=1}^{n} Y_j^2$, and for all $i \in \{1, \ldots, [(n-1)/2]_e\}$,

$$Y'V_iY = Y'C_{n,i}Y = 2\left(\sum_{j=1}^{n-i} Y_jY_{j+i} + \sum_{j=0}^{i-1} Y_{i-j}Y_{n-j}\right)$$

and, if n is even,

$$Y'C_{n,(n/2)}Y = 2\sum_{j=1}^{n/2} Y_jY_{(n/2)+j} = \sum_{j=1}^{n/2} Y_jY_{(n/2)+j} + \sum_{j=0}^{(n/2)-1} Y_{(n/2)-j}Y_{n-j}.$$

Then

$$\widehat{l'\theta} = \frac{l_0}{n}\sum_{j=1}^{n} Y_j^2 + \sum_{i=0}^{[n/2]_e} \frac{l_i}{n}\left(\sum_{j=1}^{n-i} Y_jY_{j+i} + \sum_{j=0}^{i-1} Y_{i-j}Y_{n-j}\right). \qquad \square$$

Proposition 4. *For the complete Toeplitz-skewcirculant model, the optimal unbiased estimator of $l'\theta$ for all $l \in \mathbb{R}^{r+1}$ is*

$$\widehat{l'\theta} = \frac{l_0}{n}\sum_{j=1}^{n} Y_j^2 + \sum_{i=0}^{[(n-1)/2]_e} \frac{l_i}{n}\left(\sum_{j=1}^{n-i} Y_jY_{j+i} - \sum_{j=0}^{i-1} Y_{i-j}Y_{n-j}\right). \tag{15}$$

Proof. For complete Toeplitz-skewcirculant model, the optimal unbiased estimator $Y'AY$ of $l'\theta$ is such that $A = \lambda_0 I_n + \sum_{i=1}^{[(n-1)/2]_e} \lambda_i D_{n,i}$ ($\mathscr{D}_n$ is a special Jordan algebra). Moreover

$$\mathbb{E}(Y'AY) = \mathrm{Tr}\left(A\left(\theta_0 I_n + \sum_{i=1}^{[(n-1)/2]_e} \theta_i D_{n,i}\right)\right).$$

Then $\mathbb{E}(Y'AY) = l'\theta$ for all $\theta \in \Theta$ if and only if $\mathrm{Tr}(A) = l_0$ and for all $i \in \{1, \ldots, [(n-1)/2]_e\}$ $\mathrm{Tr}(AD_{n,i}) = l_i$. By considering equalities (8)–(13), the unbiased condition leads to $\lambda_0 = l_0/n$ and, for all $i \in \{1, \ldots, [(n-1)/2]_e\}$, $\lambda_i = l_i/2n$. Moreover $Y'V_0Y = Y'I_nY = \sum_{j=1}^{n} Y_j^2$ and

$$Y'V_iY = Y'D_{n,i}Y = 2\left(\sum_{j=1}^{n-i} Y_jY_{j+i} - \sum_{j=0}^{i-1} Y_{i-j}Y_{n-j}\right).$$

Then

$$\widehat{l'\theta} = \frac{l_0}{n}\sum_{j=1}^{n} Y_j^2 + \sum_{i=0}^{[(n-1)/2]_e} \frac{l_i}{n}\left(\sum_{j=1}^{n-i} Y_jY_{j+i} - \sum_{j=0}^{i-1} Y_{i-j}Y_{n-j}\right). \qquad \square$$

These are the empirical circulant and skewcirculant estimators.

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